Lecture 15: Lovász Local Lemma

Introduction

- Let $\mathbb{B}_1, \ldots, \mathbb{B}_n$ be indicator variables for bad events in an experiment
- Suppose each bad event is unlikely, that is $\mathbb{P}[\mathbb{B}_i] \leqslant p < 1$, for all $i \in \{1, \ldots, n\}$
- Our goal is to avoid all the bad events
- Observe that if $\mathbb{P}\left[\overline{\mathbb{B}_1}, \ldots, \overline{\mathbb{B}_n}\right] > 0$ then there exists a way to avoid all the bad events simultaneously
- Suppose that the events $\{\mathbb{B}_1, \ldots, \mathbb{B}_n\}$ are independent.
- Then, it is easy to see that

$$\mathbb{P}\left[\overline{\mathbb{B}_1}, \ldots, \overline{\mathbb{B}_n}\right] \geqslant (1-p)^n > 0$$

- Lovász Local Lemma shall help us conclude the same even in the presence of "limited dependence" between the events

The Statement

Theorem

Let $(\mathbb{B}_1, \ldots, \mathbb{B}_n)$ *be a set of bad events. For each* $\mathbb{B}_i$*, where* $i \in \{1, \ldots, n\}$*, we have* $\mathbb{P}[\mathbb{B}_i] \leqslant p$ *and each event* $\mathbb{B}_i$ *depends of at most* d *other bad events. If* $\mathrm{e}p(d+1) \leqslant 1$ *then*

$$\mathbb{P}\left[\overline{\mathbb{B}_1}, \ldots, \overline{\mathbb{B}_n}\right] \geqslant \left(1 - \frac{1}{d+1}\right)^n > 0$$

The condition is also stated sometimes as $4pd \leqslant 1$ instead of $\mathrm{e}p(d+1) \leqslant 1$.

Application: k-SAT I

- Let Φ be a k-SAT formula such that each variable occurs in at most $2^{k-2}/k$ different clauses
- **Experiment.** Let $\mathbb{X}_i$ be an independent uniform random variable that assigns he variable x_i a value from $\{\text{true}, \text{false}\}$
- **Bad Events.** For the j-th clause we have the bad event $\mathbb{B}_j$ that is the indicator variable for the event: The j-th clause is not satisfied
- **Probability of a Bad Event.** For any j, note that

$$\mathbb{P}\left[\mathbb{B}_j\right] \leqslant \frac{1}{2^k}$$

Because there is at most one assignment of the variables in the clause that makes it false.

Application: k-SAT II

- **Dependence.** Note that the j-th clause has k literals. The variable associated with any literal occurs in $2^{k-2}/k$ different clauses. So, the bad event $\mathbb{B}_j$ can depend on at most $d = k \cdot (2^{k-2}/k) = 2^{k-2}$ other different bad events.
- **Conclusion.** Note that $4pd = 1$, so Lovász Local Lemma implies that there exists an assignment that satisfies all the clauses in the formula simultaneously
- Intuitively, this result states that if each variable is sufficiently localized in influence then formulas have satisfiable assignments. Noe that the probability p of each bad event does not depend on the overall problem-instance size (i.e., the total number of variables)

Application: Vertex Coloring

- Let G be a graph with degree at most Δ
- **Experiment.** Let $\mathbb{X}_v$ be the random variable that represents the color of the vertex $v \in V(G)$. Let $\mathbb{X}_v$ be a color chosen uniformly (and independently) at random from the set $\{1, \ldots, C\}$.
- **Bad Event.** For every edge $e \in E(G)$, we have a bad event $\mathbb{B}_e$ that is the indicator variable for both its vertices receiving identical colors
- **Probability of the Bad Event.** Note that $\mathbb{P}\left[\mathbb{B}_e\right] = \frac{1}{C}$
- **Dependence.** Note that the event $\mathbb{B}_e$ does not depend on any other event $\mathbb{B}_{e'}$ if the edges e and e' do not share a common vertex. So, the event $\mathbb{B}_e$ depends on at most $2(\Delta - 1)$ other bad events.
- **Conclusion.** A valid coloring exists if $4pd \leqslant 1$, i.e., $C \geqslant 8(\Delta - 1)$

Application: Vertex Coloring (Bad Bound)

- Let G be a graph with degree at most Δ
- **Experiment.** Let $\mathbb{X}_v$ be the random variable that represents the color of the vertex $v \in V(G)$. Let $\mathbb{X}_v$ be a color chosen uniformly (and independently) at random from the set $\{1, \dots, C\}$.
- **Bad Event.** For every vertex $v \in V(G)$, we have a bad event $\mathbb{B}_v$ that is the indicator variable for one of v's neighbors receives the same color as v.
- **Probability of the Bad Event.** Note that $\mathbb{P}[\mathbb{B}_v] \leqslant 1 - \left(1 - \frac{1}{C}\right)^{\Delta}$
- **Dependence.** Note that the event $\mathbb{B}_v$ does not depend on any other event $\mathbb{B}_{v'}$ if the sets $\{v\} \cup N(v)$ and $\{v'\} \cup N(v')$ do not intersect. So, the event $\mathbb{B}_v$ depends on at most $\Delta + \Delta(\Delta - 1) = \Delta^2$ other bad events
- **Conclusion.** A valid coloring exiss if $4pd \leqslant 1$, i.e., $C \geqslant$???

Proof of Lovász Local Lemma

Claim

Let $S \subseteq \{1, \ldots, n\}$. *Then, we have:*

$$\mathbb{P}\left[\mathbb{B}_i \,\middle|\, \bigwedge_{k \in S} \overline{\mathbb{B}_k}\right] \leqslant \frac{1}{d+1}$$

Assuming this claim, it is easy to prove the Lovász Local Lemma

$$\mathbb{P}\left[\bigwedge_{i=}^{n} \overline{\mathbb{B}_i}\right] = \prod_{i=1}^{n} \mathbb{P}\left[\overline{\mathbb{B}_i} \,\middle|\, \bigwedge_{k<i} \overline{\mathbb{B}_k}\right]$$

$$\geqslant \prod_{i=1}^{n}\left(1 - \frac{1}{d+1}\right) = \left(1 - \frac{1}{d+1}\right)^n > 0$$

Proof of the Claim I

- We shall proceed by induction on $|S|$
- **Base Case.** If $|S| = 0$, the the claim holds, because

$$\mathbb{P}\left[\mathbb{B}_i \,\middle|\, \bigwedge_{k \in S} \overline{\mathbb{B}_k}\right] = \mathbb{P}\left[\mathbb{B}_i\right] \leqslant p \leqslant \frac{1}{\mathrm{e}(d+1)} \leqslant \frac{1}{d+1}$$

- **Inductive Hypothesis.** Assume that the claim holds for all $|S| < t$
- **Induction.** We shall now prove the claim for $|C| = t$. Suppose D_i be the set of all j such that the bad event $\mathbb{A}_i$ (possibly) depends on the bad event $\mathbb{A}_j$
- **Easy Case.** Suppose $S_i \cap D_i = \emptyset$. This is an easy case because

$$\mathbb{P}\left[\mathbb{B}_i \,\middle|\, \bigwedge_{k \in S} \overline{\mathbb{B}_k}\right] = \mathbb{P}\left[\mathbb{B}_i\right] \leqslant p \leqslant \frac{1}{\mathrm{e}(d+1)} \leqslant \frac{1}{d+1}$$

Proof of the Claim II

- **Remaining Case.** Suppose $S \cap D_i \neq \emptyset$.

$$\mathbb{P}\left[\mathbb{B}_i \,\middle|\, \bigwedge_{k \in S} \overline{\mathbb{B}_k}\right] = \mathbb{P}\left[\mathbb{B}_i \,\middle|\, \bigwedge_{k \in S \cap D_i} \overline{\mathbb{B}_k}, \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}\right]$$

$$= \frac{\mathbb{P}\left[\mathbb{B}_i, \bigwedge_{k \in S \cap D_i} \overline{\mathbb{B}_k} \,\middle|\, \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}\right]}{\mathbb{P}\left[\bigwedge_{k \in S \cap D_i} \overline{\mathbb{B}_k} \,\middle|\, \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}\right]}$$

$$\leqslant \frac{\mathbb{P}\left[\mathbb{B}_i \,\middle|\, \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}\right]}{\mathbb{P}\left[\bigwedge_{k \in S \cap D_i} \overline{\mathbb{B}_k} \,\middle|\, \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}\right]}$$

$$= \frac{\mathbb{P}\left[\mathbb{B}_i\right]}{\mathbb{P}\left[\bigwedge_{k \in S \cap D_i} \overline{\mathbb{B}_k} \,\middle|\, \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}\right]}$$

- Our goal now is to lower-bound the denominator

Proof of the Claim III

- Suppose $S \cap D_i = \{i_1, \ldots, i_z\}$
- Using the chain rule, we can write the denominator

$$\mathbb{P}\left[\bigwedge_{k \in S \cap D_i} \overline{\mathbb{B}_k} \middle| \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}\right]$$

as follows

$$\prod_{\ell=1}^{z} \mathbb{P}\left[\overline{\mathbb{B}_{i_\ell}} \middle| \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}, \bigwedge_{k' \in \{i_1, \ldots, i_{\ell-1}\}} \overline{\mathbb{B}_{k'}}\right]$$

- Note that each probability term is conditioned on $< t$ bad events. So, we can apply the induction hypothesis. We get

$$\prod_{\ell=1}^{z} \mathbb{P}\left[\overline{\mathbb{B}_{i_\ell}} \,\middle|\, \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}, \bigwedge_{k' \in \{i_1, \ldots, i_{\ell-1}\}} \overline{\mathbb{B}_{k'}}\right] \geqslant \prod_{\ell=1}^{z}\left(1 - \frac{1}{d+1}\right)$$
$$= \left(1 - \frac{1}{d+1}\right)^z$$
$$\geqslant \left(1 - \frac{1}{d+1}\right)^d$$
$$\geqslant \frac{1}{\mathrm{e}}$$

Proof of the Claim V

- Our goal of lower-bounding the denominator is complete. Let us return to our original expression

$$\mathbb{P}\left[\mathbb{B}_i \,\middle|\, \bigwedge_{k\in S} \overline{\mathbb{B}_k}\right] \leqslant \frac{\mathbb{P}\left[\mathbb{B}_i\right]}{\mathbb{P}\left[\bigwedge_{k\in S\cap D_i} \overline{\mathbb{B}_k} \,\middle|\, \bigwedge_{k\in S\setminus D_i} \overline{\mathbb{B}_k}\right]}$$

$$\leqslant \mathrm{e}\mathbb{P}\left[\mathbb{B}_i\right] \leqslant \frac{1}{d+1}$$

- This completes the proof by induction
- We shall prove a more general result in the next lecture